Beyond the Unit Root Question: A Bounds Approach to Inference Using the Long Run Multiplier

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Quantitative Methods Reading Group

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Overview

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1. Basic Concepts
   - Time Series
   - Unit Roots
   - Spurious Regression
   - Equilibrium
   - Error Correction
   - Cointegration
2. Unit Root Tests
3. The Practical Problems of Pre-Testing
4. The Test We Want
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Big Ideas

- Uncertainty is endemic to applied time series analysis.
- This uncertainty is not reflected in conventional approaches.
- The bounds approach resolves this problem.

Basic Concepts - Time Series Analysis

Time series analysis - the application of statistical tools to time series data for the purposes of hypothesis testing and inference.

Time series data - chronological sequences of observations produced by *regularly* and *repeatedly* measuring some characteristic(s) of the same phenomenon over time.

We often assume our data are randomly generated. When we say variables are *independent and identically distributed random variables* (IID) we are making assumptions about the data.

- The observations are independent of one another.
- The observations have no meaningful order.

Time series data are not IID.

- The observations are not necessarily independent.
- The observations do have a meaningful order.

Basic Concepts - Unit Roots

Including lags of X and Y changes the way we interpret OLS regression models. We use the **Long Run Multiplier (LRM)**.

In a *static* model the coefficient is equivalent to the effect.

$$y_t = \alpha_0 + \beta x_t + u_t \quad \text{The effect of X is } \beta$$

In a *dynamic* model the effect of X on Y is given by the LRM.

The effect of X propagates into Y over time.

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta x_t + u_t \quad \text{The effect of X is } \frac{\beta}{1-\alpha_1}$$

The shock at t is discounted in future periods because $\alpha_1 < 1$.

Consider the following first-order autoregression.

$$
\begin{aligned}
y_t &= \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t; \qquad y_{t-1} = \alpha_0 + \alpha_1 y_{t-2} + \varepsilon_{t-1} \\
&= \alpha_0 + \alpha_1(\alpha_0 + \alpha_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\
&= \alpha_0 + \alpha_1\alpha_0 + \alpha_1^2 y_{t-2} + \alpha_1\varepsilon_{t-1} + \varepsilon_t \\
&= \alpha_0 + \alpha_1\alpha_0 + \alpha_1^2(\alpha_0 + \alpha_1 y_{t-3} + \varepsilon_{t-2}) + \alpha_1\varepsilon_{t-1} + \varepsilon_t \\
&= \alpha_0 + \alpha_1\alpha_0 + \alpha_1^2\alpha_0 + \alpha_1^3 y_{t-3} + \alpha_1^2\varepsilon_{t-2} + \alpha_1\varepsilon_{t-1} + \varepsilon_t \\
&= \alpha_0(1 + \alpha_1 + \alpha_1^2 + \ldots + \alpha_1^{t-1}) + \alpha_1^t y_0 + \sum_{0}^{t-1} \alpha_1^i \varepsilon_{t-i} \\
&= \alpha_1^t y_0 + \alpha_0 \sum_{i}^{t-1} \alpha_1^i + \sum_{i=0}^{t-1} \alpha_1^i \varepsilon_{t-i}
\end{aligned}
$$

Basic Concepts - Unit Roots

y_t is a *progressively discounted* summation of everything that happened in the past, the series exhibits exponential decay.

$$y_t = \alpha_1^t y_0 + \alpha_0 \sum_{i=0}^{t-1} \alpha_1^i + \sum_{i=0}^{t-1} \alpha_1^i \varepsilon_{t-i}$$

This *progressive discounting* property hinges on the value of α_1; time series are classified as I(0) or I(1) based on the value of α_1.

- **Stationary**: $|\alpha_1| < 1$
- **Non-Stationary** / **Unit Root** : $|\alpha_1| \geq 1$
  - $|\alpha_1| = 1$ *integrated* - shocks don't decay.
  - $|\alpha_1| > 1$ *explosive* - previous shocks grow.

A series is *integrated* of order *d*, where *d* is the number of times the series would need to be *differenced* $\Delta y_t = y_t - y_{t-1}$ for the series to be stationary; *stationary* (I(0)), *unit root* (I(1)).

Basic Concepts - Unit Roots

There are four types of non-stationarity we are likely to observe.

Consider the following, general, autoregressive equation:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \eta t + \varepsilon_t$$

The variety of non-stationarity depends on:

- The value of the autoregressive parameter α_1, and
- The deterministic components

Two types of deterministic components impart *trending behavior*, cause the series Y_t to increases or decreases over time.

- **Drift:** The series has a constant, $\alpha_0 \neq 0$.
- **Trend:** The series is a linear function of time, $\eta \neq 0$.

The values of α_1, α_0, and η determine the trajectory of the series.

Basic Concepts - Unit Roots

Combinations of α_1, α_0, and η that produce stationary processes.

If $\alpha_0 = \eta = \alpha_1 = 0$, y_t is **white noise**.

$$y_t = \varepsilon_t$$

If $\alpha_0 = \eta = 0$ and $|\alpha_1| < 1$, y_t is **stationary** and $E[Y] = 0$

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t$$

If $\eta = 0$, $\alpha_0 \neq 0$ and $|\alpha_1| < 1$, y_t is **stationary** and $E[Y] = \alpha$

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$$

Basic Concepts - Unit Roots

If $\alpha_0 = \eta = 0$ and $|\alpha_1| = 1$, **random walk**.

$$y_t = y_{t-1} + \varepsilon_t$$

If $\eta = 0$, $\alpha_0 \neq 0$, and $|\alpha_1| = 1$, **random walk with drift**.

$$y_t = \alpha_0 + y_{t-1} + \varepsilon_t$$

If $\eta \neq 0$, $\alpha_0 = 0$, and $|\alpha_1| = 1$, **random walk with trend**.

$$y_t = y_{t-1} + \eta t + \varepsilon_t$$

If $\eta \neq 0$, $\alpha_0 \neq 0$, $|\alpha_1| = 1$, **random walk with trend and drift**.

$$y_t = \alpha_0 + y_{t-1} + \eta t + \varepsilon_t$$

Basic Concepts - Unit Roots

There is a final possibility is that is not shown above.

If $\eta \neq 0$, $\alpha_0 = 0$ or $\alpha_0 \neq 0$, and $|\alpha_1| < 1$, y_t is **trend stationary**.

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \eta t + \varepsilon_t$$

The series is increasing over time and appears to be trending.

- The series does not integrate previous shocks.
- The series is stationary around the trend, and you can model the series as a stationary process if you "control for" the trend.

You can *detrend* a variable by regressing the series on a counter / date variable, or you can include the counter / date in a regression.

The Spurious Regression Problem

Yule (1926) showed that regressions involving independent unit root processes tend to produce *spurious* or *nonsense* results.

Consider two, independent, unit root processes:

$$y_t = \alpha_1 y_{t-1} + u_t \quad \text{and} \quad x_t = \rho_1 x_{t-1} + \epsilon_t$$

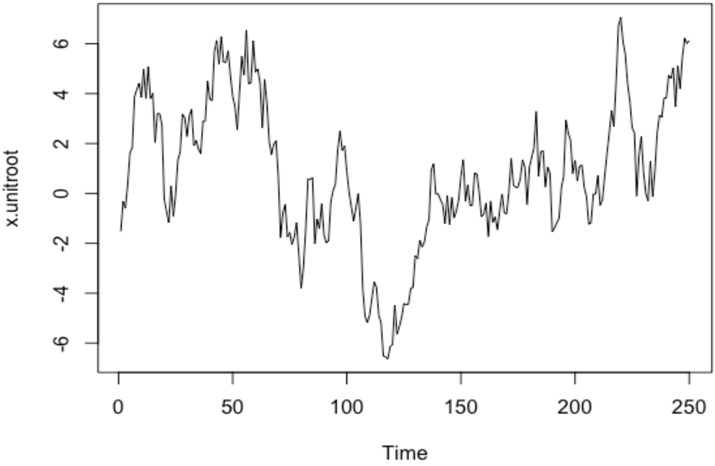
The regression of y_t on x_t will produce

- Significant t-statistic for β_{YX}
- $R^2 > 0$

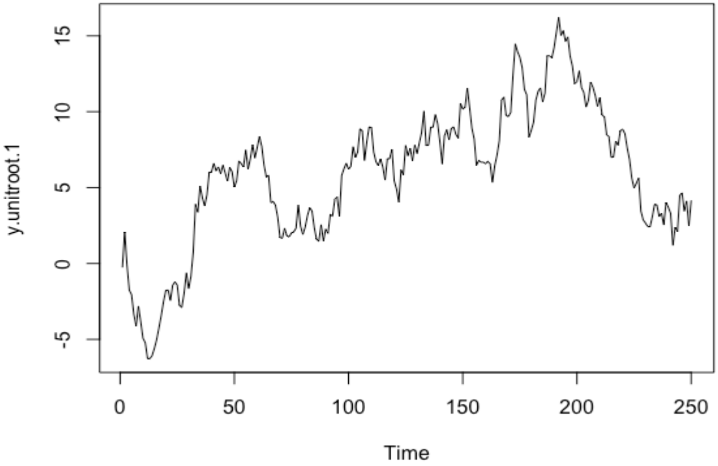
The problem gets worse as $T \to \infty$.

Both series are randomly walking. As T increases, the chances that they will randomly walk together for some period increases as well.

Basic Concepts - Spurious Regression



Basic Concepts - Spurious Regression



Basic Concepts - Spurious Regression

```
> reg.model.ur <- summary(lm(y.unitroot~x.unitroot))
> reg.model.ur

Call:
lm(formula = y.unitroot ~ x.unitroot)

Residuals:
   Min     1Q Median     3Q    Max 
-6.730 -3.732 -1.133  3.990  8.606 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) -3.39638    0.28228 -12.032  < 2e-16 ***
x.unitroot   0.51367    0.09301   5.523 8.41e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.3 on 248 degrees of freedom
Multiple R-squared:  0.1095,	Adjusted R-squared:  0.1059 
F-statistic:  30.5 on 1 and 248 DF,  p-value: 8.409e-08
```

Equilibrium

An **equilibrium state** is:

> *A state in which there is no inherent tendency to change. A disequilibrium is any situation that is not an equilibrium and hence characterizes a state that contains the seeds of its own destruction.* (Banerjee et al. 1987)

As social scientists, it is useful to think about equilibria as relationships among variables, or as equilibrium relationships.

In time series, a **long run equilibrium** is:

> The relationship to which a system converges to over time.

Basic Concepts - Equilibrium

When we estimate relationships among variables, we assume our sample relationship is representative of the equilibrium relationship.

For a variable Y, with respect to another variable X, there are three types of equilibrium relationships that could exist:

1. A *stationary unconditional* equilibrium
2. A *stationary conditional* equilibrium
3. A *cointegrating* equilibrium

Basic Concepts - Equilibrium

Consider the bivariate model:

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \beta_1 X_t + \beta_2 X_{t-1} + \varepsilon_t$$

A **stationary unconditional equilibrium** exists when:

- Y_t is a stationary process, $\alpha_1 < 1$.
- Y_t has it's own equilibrium, α_0.
- Y_t is not related to X_t, $\beta_1 = \beta_2 = 0$

Because $\beta_1 = \beta_2 = 0$, the value of Y_t does not depend on, is not *conditioned on*, X_t or its lags. It returns to its long run mean.

To illustrate, consider a unit root process (X) and a stationary process (Y), two super-interesting political economy time series.

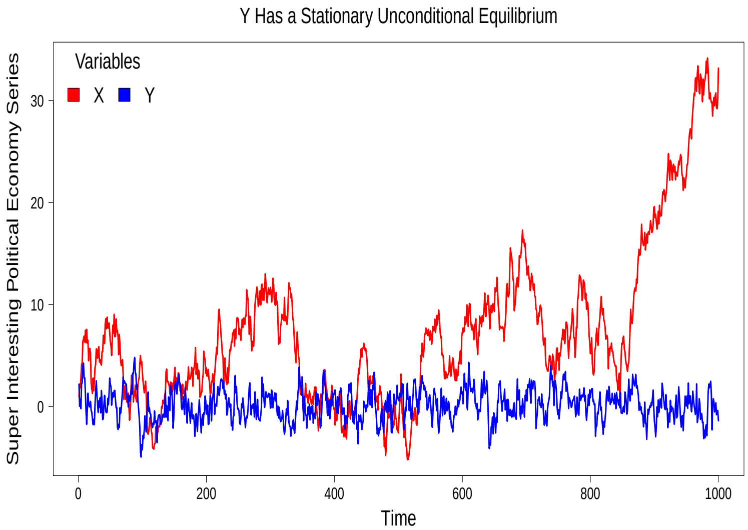
Basic Concepts - Equilibrium

```
# Generate X
  gen.x1 <- function(n, rho.x1){

    e.x1 <- rnorm(n, mean = 0, sd = 1)

    x1 <- double(n)

    x1[1] <- rnorm(1)

    for(i in 2:n){

      x1[i] <- rho.x1 * x1[i-1] + e.x1[i]

    }

    return(x1)
  }

# Generate Y
  gen.y <- function(n, rho.y, co, ct,
                    b1, b2,
                    x1, t){

    e.y <- rnorm(n, mean = 0, sd = 1)

    y <- double(n)

    y[1] <- rnorm(1)

    for(i in 2:n){

      y[i] <- co + rho.y * y[i-1] + b1 * x1[i] + b2 * x1[i-1] + ct*t[i] + e.y[i]

    }

    return(y)

  }
```

Basic Concepts - Equilibrium

```
# Inputs
  n <- 1000

  rho.x1 <- 1
  rho.y <- .7

  co <- 0
  ct <- 0

  b1 <- 0
  b2 <- 0

  set.seed(19852015)

# Generate Variables
  t <- gen.t(n = n)

  x1 <- gen.x1 (n = n, rho.x1 = rho.x1)

  y <- gen.y(n = n, rho.y = rho.y,
             co = co, ct = ct, b1 = b1, b2 = b2,
             x1, t)

 # Create time series object data frame
   ts.sim <- data.frame(cbind(t,y,x1))

 # Plot X and Y
   par(oma = c(.7,1.5,1,0.5))
   plot(y = ts.sim[,"x1"], x = ts.sim[,"t"], type = "l",
        yaxt = "n", xaxt = "n",
        ylab = "", xlab = "", cex.axis = 1.5,
        lwd = 3, col = "red")
   lines(y = ts.sim$y, x = ts.sim$t, lwd = 3, col = "blue")
   axis(2, las = 2, cex.axis = 1.5)
   axis(1, las = 1, cex.axis = 1.5)

   mtext(text = "My Fake Variables", side = 2, line = 3.5, cex = 2)
   mtext(text = "Time", side = 1, line = 3, cex = 2)
   mtext(text = "Y Has a Stationary Unconditional Equilibrium", side = 3, line = 1.25, cex = 2.0)
   legend("topleft",cex = 2.0, bty="n",
          legend = c("X","Y"),
          ncol=2,
          fill = c("red", "blue"),
          title= "Variables")
```

Basic Concepts - Equilibrium



Basic Concepts - Equilibrium

Next, imagine that our two super-interesting political economy time series (X) and (Y) are stationary processes that are related.

Again we have the bivariate model

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \beta_1 X_t + \beta_2 X_{t-1} + \varepsilon_t$$

A **stationary conditional equilibrium** exists when:

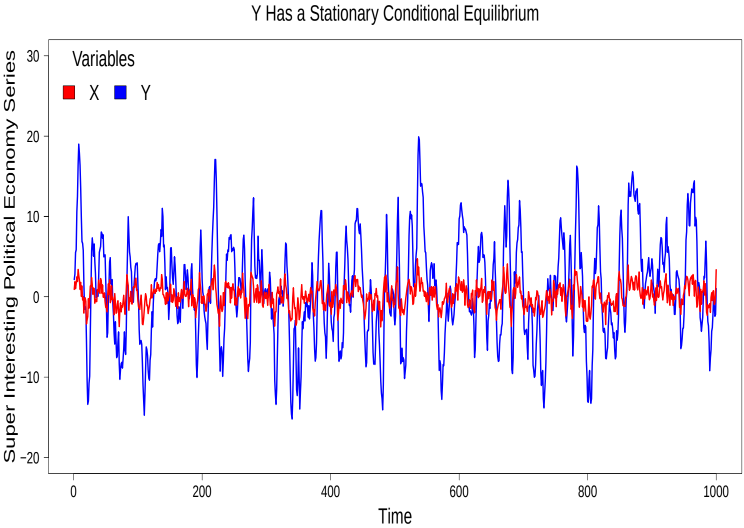
- Y_t still has its own equilibrium, α_0.
- But Y_t is related to X_t:
  - $\beta_1 \neq 0$,
  - $\beta_2 \neq 0$, or
  - $\beta_1 \neq 0$ and $\beta_2 \neq 0$

This is the type of relationship we model with an autoregressive distributed lag (ADL) model or one of its restricted specifications.

Basic Concepts - Equilibrium

```r
# Conditional Stationary Equilibrium

  # Inputs
     n <- 1000

     rho.x1 <- .7
     rho.y <- .7

     co <- 0
     ct <- 0

     b1 <- 1
     b2 <- 1

     set.seed(19852015)

  # Generate Variables
     t <- gen.t(n = n)

     x1 <- gen.x1 (n = n, rho.x1 = rho.x1)

     y <- gen.y(n = n, rho.y = rho.y,
                co = co, ct = ct, b1 = b1, b2 = b2,
                x1, t)

  # Create time series object data frame
     ts.sim <- data.frame(cbind(t,y,x1))

  # Plot X and Y
     par(oma = c(.7,1.5,1,0.5))
     plot(y = ts.sim[,"y"], x = ts.sim[,"t"], type = "l",
          yaxt = "n", xaxt = "n",
          ylab = "", xlab = "", cex.axis = 1.5,
          lwd = 3, col = "blue",
          ylim = c(-20,30))
     lines(y = ts.sim$x, x = ts.sim$t, lwd = 3, col = "red")
     axis(2, las = 2, cex.axis = 1.5)
     axis(1, las = 1, cex.axis = 1.5)

     mtext(text = "Super Interesting Political Economy Series", side = 2, line = 3.5, cex = 2)
     mtext(text = "Time", side = 1, line = 3, cex = 2)
     mtext(text = "Y Has a Stationary Conditional Equilibrium", side = 3, line = 1.25, cex = 2.0)
     legend("topleft",cex = 2.0, bty="n",
            legend = c("X","Y"),
            ncol=2,
            fill = c("red", "blue"),
            title= "Variables")
```

Basic Concepts - Equilibrium



Basic Concepts - Equilibrium

In both cases, Y has a **stationary equilibrium** it returns to, α_0.

This equilibrium is the stationary point toward which the series tends to return to whenever it moves away. The series is *at its equilibrium* when the following condition is met

$$y_t = \alpha_0$$

In most time periods, however, the series (Y) will not be at its equilibrium. It will be *away from its equilibrium*:

$$y_t = \alpha_0 + z_t$$

Where z_t may be called equilibrium error. Stationary series have stationary equilibria and they tend to "prefer" smaller values of z_t.

The rate at which z_t is reduced, which is the rate that y_t returns to its equilibrium (α_0), is the *error correction rate*.

Error Correction

All stationary time series have an error correction rate.

Consider our models from above:

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \beta_1 X_t + \beta_2 X_{t-1} + \varepsilon_t$$

The error correction rate is calculated:

$$\alpha_1 - 1$$

This formula highlights the relationship between the persistance we observe in a time series and the observed error correction rate.

- If α_1 is closer to 1, the error correction rate is slower.
- If α_1 is closer to 0, the error correction rate is faster.
- If $\alpha_1 = 0$, the series returns to its equilibrium in every period.

Cointegration

An *integrated* series does not have a stationary equilibrium.

- There is no constant α_0 that it tends to cross.
- The error z_t is theoretically infinite, there is no correction.
- Any sample will have a mean. It is possible that a series will cross this mean due to random chance but the mean of the series is not an equilibrium, it is incidental to the sample.

It is possible for an integrated series to have an equilibrium relationship with respect to another integrated series.

We call this phenomenon, *cointegration*

Two series are said to be *co* - *integrated* if both series are integrated of the same order I(d) but there is a linear combination of the series that produces a stationary I(0) process.

Basic Concepts - Cointegration

Consider two random walk processes:

$$Y_t = Y_{t-1} + \varepsilon_t \qquad \text{And} \qquad X_t = X_{t-1} + \varepsilon_t$$

These series are combined

$$Y_t = \beta_1 X_t + \nu_t \qquad \text{such that} \qquad \nu_t = Y_t - \beta_1 X_t$$

If ν_t is an I(0) process, the series are *cointegrated*.

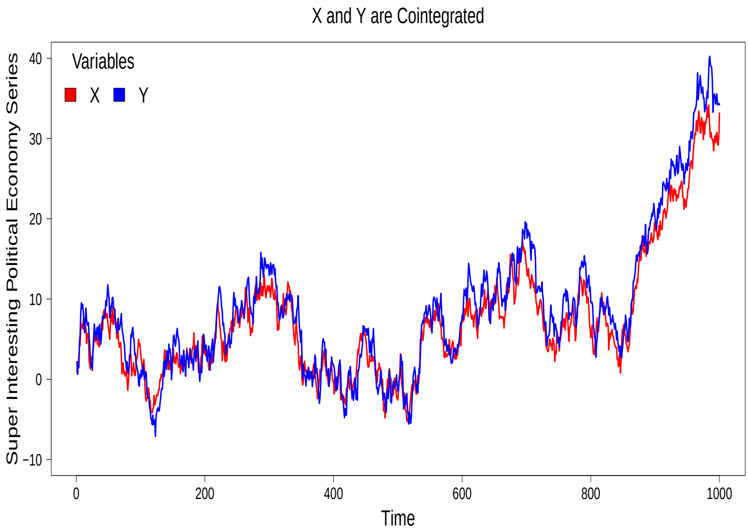
The ν_t series doesn't drift far from zero because X and Y never drift too far apart. They wander, but they wander together.

This final variety of equilibrium relationship is referred to as a **Cointegrating Equilibrium**. X and Y do not have their own equilibriums but they are in equilibrium with one another.

Basic Concepts - Cointegration

```
# Inputs
   n <- 1000

   rho.x1 <- 1
   rho.y <- .7

   co <- 0
   ct <- 0

   b1 <- .25
   b2 <- .1

   set.seed(19852015)

# Generate Variables
   t <- gen.t(n = n)

   x1 <- gen.x1 (n = n, rho.x1 = rho.x1)

   y <- gen.y(n = n, rho.y = rho.y,
              co = co, ct = ct, b1 = b1, b2 = b2,
              x1, t)

# Create time series object data frame
   ts.sim <- data.frame(cbind(t,y,x1))

   # Plot X and Y
   par(oma = c(.7,1.5,1,0.5))
   plot(y = ts.sim[,"x1"], x = ts.sim[,"t"], type = "l",
        yaxt = "n", xaxt = "n",
        ylab = "", xlab = "", cex.axis = 1.5,
        lwd = 3, col = "red",
        ylim = c(-10,40))
   lines(y = ts.sim$y, x = ts.sim$t, lwd = 3, col = "blue")
   axis(2, las = 2, cex.axis = 1.5)
   axis(1, las = 1, cex.axis = 1.5)

   mtext(text = "Super Interesting Political Economy Series", side = 2, line = 3.5, cex = 2)
   mtext(text = "Time", side = 1, line = 3, cex = 2)
   mtext(text = "X and Y are Cointegrated", side = 3, line = 1.25, cex = 2.0)
   legend("topleft",cex = 2.0, bty="n",
          legend = c("X","Y"),
          ncol=2,
          fill = c("red", "blue"),
          title= "Variables")
```

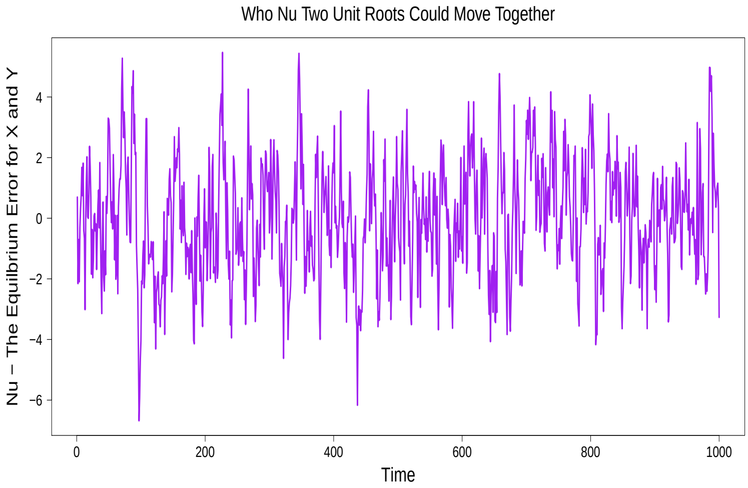
Basic Concepts - Cointegration



Basic Concepts - Cointegration

```
> # The residual nu series
>      ecreg <- lm(y ~ x1, data = ts.sim)
>
>      summary(ecreg)

Call:
lm(formula = y ~ x1, data = ts.sim)

Residuals:
    Min      1Q  Median      3Q     Max
-6.6785 -1.2818 -0.0256  1.3424  5.4671

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.420086   0.083049   5.058 5.03e-07 ***
x1          1.118516   0.007721 144.863  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.9 on 998 degrees of freedom
Multiple R-squared:  0.9546,    Adjusted R-squared:  0.9546
F-statistic: 2.099e+04 on 1 and 998 DF,  p-value: < 2.2e-16

>
>      nu <- ecreg$residuals
>
>      par(oma = c(.7,1.5,1,0.5))
>      plot(y = nu, x = ts.sim[,"t"], type = "l",
+           yaxt = "n", xaxt = "n",
+           ylab = "", xlab = "", cex.axis = 1.5,
+           lwd = 3, col = "purple")
>      axis(2, las = 2, cex.axis = 1.5)
>      axis(1, las = 1, cex.axis = 1.5)
>      mtext(text = "Nu - The Equilbrium Error for X and Y", side = 2, line = 3.5, cex = 2)
>      mtext(text = "Time", side = 1, line = 3, cex = 2)
>      mtext(text = "Who Nu Two Unit Roots Could Move Together", side = 3, line = 1.25, cex = 2.0)
```

Basic Concepts - Cointegration



Basic Concepts - Cointegration

Cointegration is a "rare phenomenon." It is rare that there is a constant (β_1) that will allow ν to be a stationary series.

> Engle, Robert and C.W.J. Granger. 1987. "Cointegration and Error Correction: Representation, Estimation, and Testing." *Econometrica* 55(2): 251-76.

That said, there are some instances where theory would cause us to expect to observe cointegrating equilibria.

- The relationship between short and long term interest rates
- Capital appropriations and expenditures
- Prices of the same commodity in different markets

Of course, it is not sufficient to just have a conjecture about the presence of cointegration. We will learn how to:

1. Test for the presence of cointegrating equilibria, and
2. Use error-correction models to model these equilibria.

How Can We Identify Unit Roots?

Question: Is my time series stationary or non-stationary?

- If I have trend or drift, it's easy to tell.
- If not, Y might be a unit root or just strongly autoregressive.
- I may have a trend stationary series.

For simplicity, consider an AR(1) process

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t$$

We want to know whether $\alpha_1 \leq 1.0$, why not use $t_{\alpha_1} = \frac{\alpha_1}{SE(\alpha_1)}$?

- Classical hypothesis tests won't work. If the true DGP is a unit root, the central limit theorem does not apply.
- We can calculate a *t*-test but Dickey and Fuller (1979) show that the distribution for this statistic is not *t*-distributed.

How Can We Identify Unit Roots?

The Dickey-Fuller Test

Dickey and Fuller (1979) solve this problem by subtracting y_{t-1} from both sides, writing the first order process in differences.

$$\begin{aligned} y_t - y_{t-1} &= \alpha_1 y_{t-1} - y_{t-1} + \varepsilon_t \\ \Delta y_t &= (\alpha_1 - 1) y_{t-1} + \varepsilon_t \\ \Delta y_t &= \gamma y_{t-1} + \varepsilon_t \end{aligned}$$

In the final equation, $\gamma = \alpha_1 - 1$. Testing the null hypothesis $\gamma = 0$ is the same as testing the null hypothesis $\alpha_1 = 1$.

- If $\alpha_1 = 1$, then $\gamma = 0$. This means y_{t-1} cancels out; Δy_t are just random shocks because y_t is a random walk.
- If $\alpha_1 < 1$, then $\gamma < 1$, and y_{t-1} is stationary.

How Can We Identify Unit Roots?

We estimate the Dickey Fuller regression using OLS:

$$\Delta y_t = \hat{\gamma} y_{t-1} + \varepsilon_t$$

A *t-like* test on $\hat{\gamma}$ can tell us if $\gamma = 0$ but the central limit theorem does not apply to this test statistic and t_γ is not t distributed.

The test statistic is **Dickey-Fuller** distributed. The distribution is predictable. Dickey and Fuller (1979) tabulated the critical values under the null that the series is non-stationary.

This is why we say the test statistic is *Dickey-Fuller* distributed.

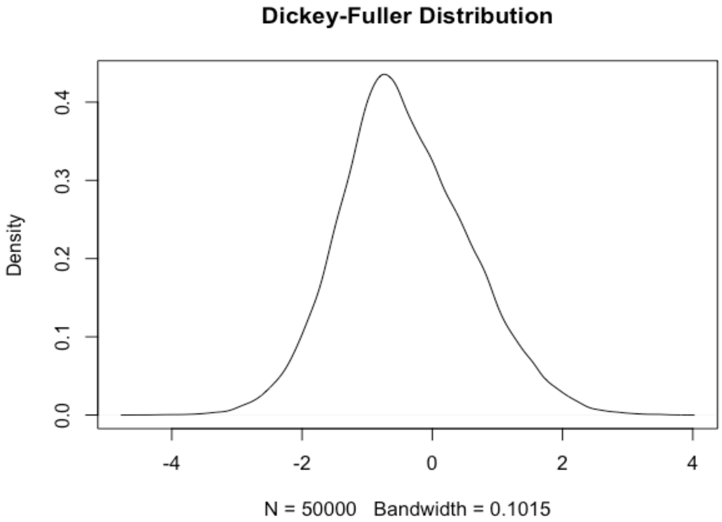
Note: $\hat{\gamma}$ will, typically, be negative, so the test statistic (t_{DF}) will also be negative. To reject the *unit root null*, $t_{DF} < t^*_{DF}$.

- If we **reject** the null, we conclude y_t is **stationary**.
- If we **fail to reject** the null, y_t contains a **unit root**.

How Can We Identify Unit Roots?

```
> dfuller.values <- function(n,nsim,rho.x){
+   out <- vector(mode="list", length=nsim)
+   for(m in 1:nsim){
+
+   x <- gen.x(n,rho.x=rho.x)
+   x.diff <- x - tslag(x)
+   lagx <- tslag(x)
+
+   simdf <- data.frame(x,x.diff,lagx)
+
+
+   lm.df <- summary(lm(x.diff~lagx -1 ,data=simdf))
+   t.df <- lm.df$coefficients[,3][[1]]
+   out[[m]] <- t.df
+   }
+   unlist(out)
+ }
>
> out <- dfuller.values(100,50000,1)
> plot(density(unlist(out)),main="Dickey-Fuller Distribution")
> quantile(out,c(.01,.025,.05,.1))
       1%       2.5%         5%        10%
-2.572757 -2.223459 -1.932680 -1.606744
```

How Can We Identify Unit Roots?



How Can We Identify Unit Roots?

The basic DF test is based on an Auxiliary OLS regression.

$$\Delta y_t = \hat{\gamma} y_{t-1} + \varepsilon_t$$

The actual test is a *t-like* test on the coefficient for y_{t-1}.

$$t_{DF} = \frac{\hat{\gamma}}{SE(\hat{\gamma})}$$

It's *t-like* because the statistic isn't distributed t. Otherwise inference proceeds like the t-test for $\hat{\beta}$ from an OLS model.

$$t = \frac{\hat{\beta}}{SE(\hat{\beta})}$$

Simple enough, so what's the big deal?

How Can We Identify Unit Roots?

In addition to having a non-standard distribution, the Dickey-Fuller test is very sensitive to how the auxiliary model is specified.

The auxiliary regression is an OLS regression, so it relies on the OLS assumptions. Three assumptions are particularly important:

- No autocorrelation
- The model is correctly specified.
- No heteroskedasticity

These assumptions mean that we have to be very careful about how we set up the test regression before we go to interpret the test.

Getting the test regression right turns out to be a little involved.

How Can We Identify Unit Roots?

Modified versions of the Dickey-Fuller test, and a number of alternative tests, have been proposed to classify series.

- Augmented Dickey-Fuller Test (1979) - ADF Test
- Phillips-Perron Test (1988) - PP Test
- Kwiatokowksi, Phillips, Schmidt, and Shin (1992) - KPSS Test
- Elliott, Rothenberg, and Stock (1996) - DF-GLS Test
- Schmidt, Peter and Peter Phillips (1992) - SPP LM Test
- Zivot and Andrews (2002) - Breakpoint Test
- Lo and MacKinlay (1988) - Variance Ratio Test
- Lo (1989) - Modified Rescaled Range Test
- ...

The application of these tests bedeviled by a number of problems.

The Practical Problems of Pretesting

We are often interested in long-run relationships between some (set of) independent variable(s), X, and some process, Y.

Traditional approaches have three steps:

1. Use pre-tests to identify the orders of integration and univariate properties of each time series,
2. Choose an appropriate model and hypothesis testing framework based on these results, and
3. Draw inferences from the model.

Fundamental Problem: There is uncertainty in the first step of the process that is not reflected in the steps that follow.

The Practical Problems of Pretesting

The tests commonly used to classify time series as stationary or unit root processes are notoriously unreliable.

Factors complicating unit root testing procedures:

1. The tests have low power
2. Test choice depends on knowledge of deterministics
3. Lag length and lag-truncation
4. Level of significance

Pre-tests often produce inconclusive and conflicting results.

There are no criteria we can use to arbitrate among these results.

The Practical Problems of Pretesting

Table 4: LRM-t and PSS Conditional Rejection Rates for $\rho_y = .7$ and $D = \{1, 0\}$

		$T = 50$			$T = 150$			$T = 250$			$T = 350$			$T = 1000$		
	Lag	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t
Dickey-Fuller ϕ_3	0	0.343			0.999			1.000			1.000			1.000		
Dickey-Fuller τ_τ	0	0.426	0.474	0.024	1.000			1.000			1.000			1.000		
Dickey-Fuller ϕ_2	0	0.229			0.998			1.000			1.000			1.000		
Dickey-Fuller ϕ_1	0	0.530			1.000			1.000			1.000			1.000		
Dickey-Fuller τ_μ	0	0.633	0.300	0.011	1.000			1.000			1.000			1.000		
Dickey-Fuller τ	0	0.008	0.646	0.175	0.344	1.000	0.994	0.984	1.000	1.000	1.000			1.000		
Augmented Dickey-Fuller ϕ_3	1	0.255			0.994			1.000			1.000			1.000		
Augmented Dickey-Fuller τ_τ	1	0.321	0.563	0.113	0.997	1.000	0.667	1.000			1.000			1.000		
Augmented Dickey-Fuller ϕ_2	1	0.156			0.986			1.000			1.000			1.000		
Augmented Dickey-Fuller ϕ_1	1	0.395			1.000			1.000			1.000			1.000		
Augmented Dickey-Fuller τ_μ	1	0.481	0.466	0.046	1.000			1.000			1.000			1.000		
Augmented Dickey-Fuller τ	1	0.003	0.648	0.177	0.087	1.000	0.996	0.707	1.000	1.000	0.993	1.000	1.000	1.000		
ADF GLS Constant		0.591	0.621	0.164	0.930	1.000	1.000	0.964	1.000	1.000	0.990	1.000	1.000	1.000		
ADF GLS Trend		0.249	0.627	0.152	0.938	1.000	1.000	0.991	1.000	1.000	0.998	1.000	1.000	1.000		
KPSS μ	None	0.612	0.552	0.552	0.629	1.000	1.000	0.660	1.000	1.000	0.665	1.000	1.000	0.684	1.000	1.000
KPSS τ	None	0.197	0.538	0.538	0.173	1.000	1.000	0.137	1.000	1.000	0.144	1.000	1.000	0.141	1.000	1.000
KPSS μ	Short	0.044	0.545	0.545	0.070	1.000	1.000	0.061	1.000	1.000	0.060	1.000	1.000	0.079	1.000	1.000
KPSS τ	Short	0.776	0.599	0.599	0.895	1.000	1.000	0.915	1.000	1.000	0.904	1.000	1.000	0.919	1.000	1.000
KPSS μ	Long	0.247	0.526	0.526	0.206	1.000	1.000	0.216	1.000	1.000	0.208	1.000	1.000	0.162	1.000	1.000
KPSS τ	Long	0.060	0.700	0.700	0.066	1.000	1.000	0.070	1.000	1.000	0.083	1.000	1.000	0.066	1.000	1.000
PP Z_τ Constant	Short	0.712	0.264	0.007	1.000			1.000			1.000			1.000		
PP Z_τ Trend	Short	0.593	0.324	0.012	1.000			1.000			1.000			1.000		
PP Z_τ Constant	Long	0.498	0.436	0.022	1.000			1.000			1.000			1.000		
PP Z_τ Trend	Long	0.347	0.518	0.055	0.999	1.000	0.000	1.000			1.000			1.000		
Variance Ratio		0.437	0.542	0.114	0.859	1.000	0.986	0.963	1.000	1.000	0.989	1.000	1.000	1.000		
Variance Ratio Trend		0.329	0.580	0.104	0.880	1.000	0.992	0.976	1.000	1.000	0.996	1.000	1.000	1.000		

The Practical Problems of Pretesting

Table 3: LRM-t and PSS Conditional Rejection Rates for $\rho_y = .8$ and $D = \{1, 0\}$

		$T = 50$			$T = 150$			$T = 250$			$T = 350$			$T = 1000$		
	Lag	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t
Dickey-Fuller ϕ_3	0	0.165			0.926			1.000			1.000			1.000		
Dickey-Fuller τ_τ	0	0.193	0.344	0.027	0.965	0.943	0.057	1.000			1.000			1.000		
Dickey-Fuller ϕ_2	0	0.103			0.872			1.000			1.000			1.000		
Dickey-Fuller ϕ_1	0	0.257			0.992			1.000			1.000			1.000		
Dickey-Fuller τ_μ	0	0.331	0.229	0.018	0.999	1.000	0.000	1.000			1.000			1.000		
Dickey-Fuller τ	0	0.001	0.431	0.074	0.002	0.997	0.764	0.031	1.000	0.999	0.331	1.000	1.000	1.000		
Augmented Dickey-Fuller ϕ_3	1	0.134			0.852			0.999			1.000			1.000		
Augmented Dickey-Fuller τ_τ	1	0.173	0.370	0.044	0.914	0.977	0.244	1.000			1.000			1.000		
Augmented Dickey-Fuller ϕ_2	1	0.084			0.775			0.999			1.000			1.000		
Augmented Dickey-Fuller ϕ_1	1	0.223			0.976			1.000			1.000			1.000		
Augmented Dickey-Fuller τ_μ	1	0.292	0.305	0.028	0.991	1.000	0.000	1.000			1.000			1.000		
Augmented Dickey-Fuller τ	1	0.001	0.432	0.074	0.000	0.997	0.764	0.008	1.000	0.999	0.088	1.000	1.000	1.000		
ADF GLS Constant		0.460	0.391	0.061	0.886	0.991	0.763	0.951	1.000	0.980	0.981	1.000	1.000	1.000		
ADF GLS Trend		0.175	0.402	0.059	0.814	0.984	0.667	0.976	1.000	1.000	0.997	1.000	1.000	1.000		
KPSS μ	None	0.747	0.388	0.388	0.803	0.996	0.996	0.851	1.000	1.000	0.868	1.000	1.000	0.857	1.000	1.000
KPSS τ	None	0.266	0.289	0.289	0.286	0.993	0.993	0.229	1.000	1.000	0.245	1.000	1.000	0.202	1.000	1.000
KPSS μ	Short	0.056	0.286	0.286	0.117	0.983	0.983	0.096	1.000	1.000	0.091	1.000	1.000	0.084	1.000	1.000
KPSS τ	Short	0.876	0.396	0.396	0.962	0.997	0.997	0.983	1.000	1.000	0.985	1.000	1.000	0.989	1.000	1.000
KPSS μ	Long	0.319	0.379	0.379	0.370	0.992	0.992	0.332	1.000	1.000	0.344	1.000	1.000	0.271	1.000	1.000
KPSS τ	Long	0.061	0.426	0.426	0.109	0.991	0.991	0.106	1.000	1.000	0.096	1.000	1.000	0.103	1.000	1.000
PP Z_τ Constant	Short	0.398	0.203	0.012	0.998	1.000	0.500	1.000			1.000			1.000		
PP Z_τ Trend	Short	0.311	0.245	0.012	0.997	0.333	0.000	1.000			1.000			1.000		
PP Z_τ Constant	Long	0.249	0.324	0.027	0.973	0.926	0.111	1.000			1.000			1.000		
PP Z_τ Trend	Long	0.161	0.359	0.027	0.952	0.938	0.125	1.000			1.000			1.000		
Variance Ratio		0.301	0.379	0.046	0.694	0.993	0.654	0.866	1.000	0.993	0.937	1.000	1.000	1.000		
Variance Ratio Trend		0.194	0.387	0.060	0.646	0.992	0.630	0.878	1.000	0.992	0.959	1.000	1.000	0.999	1.000	1.000

The Practical Problems of Pretesting

Table 2: LRM-t and PSS Conditional Rejection Rates for $\rho_y = .9$ and $D = \{1, 0\}$

		$T = 50$			$T = 150$			$T = 250$			$T = 350$			$T = 1000$		
	Lag	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t	Test	PSS_F	PSS_t
Dickey-Fuller ϕ_3	0	0.059	0.189	0.189	0.303	0.561	0.561	0.759	0.917	0.917	0.975	1.000	1.000	1.000		
Dickey-Fuller τ_τ	0	0.075	0.185	0.004	0.387	0.519	0.020	0.832	0.893	0.030	0.987	1.000	0.000	1.000		
Dickey-Fuller ϕ_2	0	0.036	0.203	0.008	0.234	0.595	0.039	0.643	0.941	0.090	0.950	1.000	0.140	1.000		
Dickey-Fuller ϕ_1	0	0.092	0.154	0.006	0.514	0.366	0.006	0.950	0.580	0.000	0.999	1.000	0.000	1.000		
Dickey-Fuller τ_μ	0	0.125	0.134	0.003	0.616	0.276	0.005	0.978	0.273	0.000	1.000			1.000		
Dickey-Fuller τ	0	0.359	0.125	0.011	0.965	0.029	0.000	1.000			1.000			1.000		
Augmented Dickey-Fuller ϕ_3	1	0.063	0.203	0.203	0.285	0.576	0.576	0.691	0.942	0.942	0.957	1.000	1.000	1.000		
Augmented Dickey-Fuller τ_τ	1	0.071	0.200	0.011	0.352	0.545	0.039	0.787	0.915	0.085	0.976	1.000	0.167	1.000		
Augmented Dickey-Fuller ϕ_2	1	0.036	0.210	0.012	0.209	0.606	0.058	0.582	0.947	0.213	0.913	1.000	0.460	1.000		
Augmented Dickey-Fuller ϕ_1	1	0.087	0.177	0.009	0.471	0.435	0.023	0.918	0.805	0.024	1.000			1.000		
Augmented Dickey-Fuller τ_μ	1	0.123	0.166	0.009	0.570	0.360	0.012	0.964	0.583	0.028	1.000			1.000		
Augmented Dickey-Fuller τ	1	0.349	0.151	0.014	0.956	0.114	0.000	1.000			1.000			1.000		
ADF GLS Constant		0.274	0.193	0.021	0.661	0.575	0.133	0.862	0.957	0.543	0.918	1.000	0.976	0.998	1.000	1.000
ADF GLS Trend		0.078	0.215	0.018	0.401	0.573	0.093	0.779	0.928	0.335	0.918	1.000	0.866	1.000		
KPSS μ	None	0.876	0.187	0.187	0.961	0.669	0.669	0.974	0.977	0.977	0.975	1.000	1.000	0.985	1.000	1.000
KPSS τ	None	0.422	0.135	0.135	0.524	0.607	0.607	0.443	0.955	0.955	0.467	1.000	1.000	0.376	1.000	1.000
KPSS μ	Short	0.087	0.126	0.126	0.185	0.584	0.584	0.177	0.960	0.960	0.162	1.000	1.000	0.144	1.000	1.000
KPSS τ	Short	0.957	0.210	0.210	0.996	0.679	0.679	0.999	0.978	0.978	1.000	1.000	1.000	1.000	1.000	1.000
KPSS μ	Long	0.520	0.160	0.160	0.642	0.603	0.603	0.650	0.968	0.968	0.657	1.000	1.000	0.592	1.000	1.000
KPSS τ	Long	0.101	0.198	0.198	0.232	0.543	0.543	0.226	0.956	0.956	0.223	1.000	1.000	0.205	1.000	1.000
PP Z_τ Constant	Short	0.153	0.119	0.004	0.641	0.284	0.006	0.974	0.385	0.000	1.000			1.000		
PP Z_τ Trend	Short	0.135	0.132	0.003	0.650	0.289	0.003	0.980	0.350	0.000	1.000			1.000		
PP Z_τ Constant	Long	0.098	0.174	0.003	0.444	0.480	0.025	0.862	0.877	0.043	0.987	1.000	0.000	1.000		
PP Z_τ Trend	Long	0.072	0.186	0.004	0.404	0.505	0.022	0.873	0.866	0.008	0.990	1.000	0.100	1.000		
Variance Ratio		0.159	0.181	0.020	0.404	0.606	0.117	0.612	0.948	0.433	0.754	1.000	0.846	0.966	1.000	1.000
Variance Ratio Trend		0.081	0.203	0.011	0.304	0.602	0.095	0.524	0.960	0.393	0.688	1.000	0.853	0.986	1.000	1.000

Ambiguities in testing for LRRs

Consider the ADL(1,1) model:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + e_t$$

DeBeof and Keele (2008) show that it can be expressed as:

$$\Delta y_t = \alpha_0 + \beta_0^* \Delta x_t + \alpha_1^* y_{t-1} + \beta_1^* x_{t-1} + e_t$$

This GECM can also be expressed as:

$$\Delta y_t = \alpha_0 + \beta_0^* \Delta x_t + \alpha_1^* (y_{t-1} - \lambda x_{t-1}) + e_t$$

where

- λ is the long run multiplier.
- $y_{t-1} - \lambda x_{t-1}$ measures the disequilibrium between y and x.
- α_1^* is the error correction rate.

Ambiguities in testing for LRRs

The final model is estimated as a GECM but there are important equivalencies among the parameters of the ADL and the GECM.

Impact Multipliers

- $\beta_0^* = \beta_0$
- $\alpha_1^* \lambda = \beta_1^* = \beta_0 + \beta_1$

Long Run Multiplier

- $\lambda = -\frac{\beta_1^*}{\alpha_1^*}$ in the GECM, and
- $\lambda = \frac{\beta_0 + \beta_1}{1 - \alpha_1}$

The constant (α_0) is the same in every representation.

The interpretations of all these models depend on one's ability to classify their series as stationary or non-stationary time series.

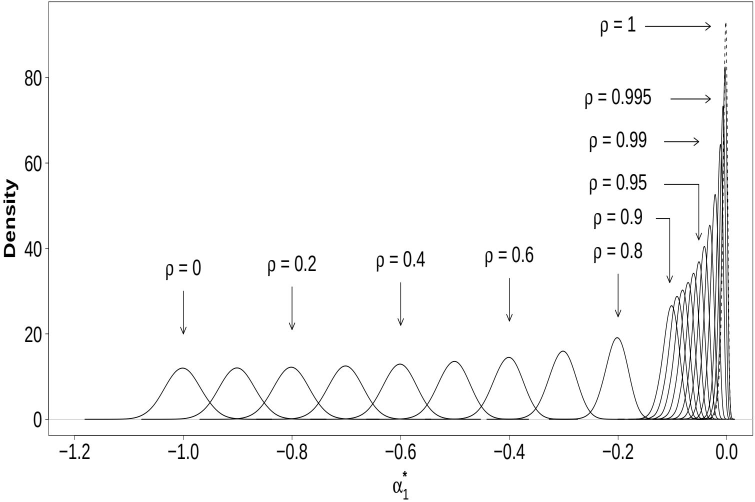
Ambiguities in testing for LRRs

Consider testing for a long run relationship using the GECM

- If we *know* we have all unit roots:
  - Inference about a LRR depends on the hypothesis that $\alpha_1^* = 0$.
  - This is a test for a cointegrating equilibrium relationship.
  - The test statistic for α_1^* is non-standard.
  - If $\alpha_1^* = 0$, λ is undefined.
- If we *know* all the data are stationary:
  - The null $\alpha_1^* = 0$ is that y_t has no equilibrium.
  - The alternative is trivial, we expect to reject the null.
  - If y is classified as stationary, $\alpha_1^* \neq 0$.
  - This result does not depend on λ or β_1^*

The value of α_1^* changes over different levels of ρ_y.

Ambiguities in testing for LRRs



Ambiguities in testing for LRRs

For a stationary series, the α_1^* coefficient is capturing the *speed of mean reversion*. It doesn't tell us something special about the relationship between X and y.

One will almost always reject the null hypothesis with respect to the error correction coefficient whether or not y is a function of X.

The Problem: Inference based on the GECM (or α_1^*) cannot discern conditional equilibria from unconditional equilibria absent **CERTAINTY** about the univariate properties of the data.

- Rejecting $\alpha_1^* = 0$ *might* imply cointegration, or not.
- α_1^* *might* measure the ECR between y and x, or not.
- Finding $\alpha_1^* = 0$ *might* mean $\lambda = \beta_1^* = 0$, or not.

The Solution: A hypothesis testing procedure that does not require certainty about the univariate properties of the data.

A Bounds Approach to Inference using the LRM

The Logic

Consider the bivariate GECM from before:

$$\Delta y_t = \alpha_0 + \beta_0^* \Delta x_t + \alpha_1^* y_{t-1} + \beta_1^* x_{t-1} + e_t$$

The LRM, $\lambda = -\frac{\beta_1^*}{\alpha_1^*}$, gives the total effect of a change in x on y.

A conditional LRR can only exist if $\lambda \neq 0$.

This requires that

- The numerator is non-zero: $\beta_1^* \neq 0$.
- The denominator is non-zero: $\alpha_1^* \neq 0$.

A test on the LRM is a simultaneous test on β_1^* and α_1^*.

A Bounds Approach to Inference using the LRM

If both X and Y are unit roots:

- Cointegration means $\alpha_1^* \neq 0$.
- This only occurs if X and Y are related, $\beta_1^* \neq 0$.
- The numerator and the denominator are non-zero.
- If they are not cointegrated, λ is undefined.

If both x and y are stationary:

- By definition, α_1^* cannot be 0.
- If X and Y are linked, $\beta_1^* \neq 0$.
- The numerator and the denominator are non-zero.
- If they are not related, $\lambda = \beta_1^* = 0$.

This logic extends to models with multiple regressors and other dynamically complete (balanced) specifications.

A Bounds Approach to Inference using the LRM

This is the test we want!

The test for the significance of the LRM in *any* dynamic regression is a test of a conditional long run relationship between X and Y, regardless of the univariate properties of the data.

There are only two possibilities:

- **Reject:** There is a LRR between X and Y.
- **Fail to Reject:** There is not a LRR between X and Y.

The Complication: The standard error for the LRM is not directly estimated in the GECM or the ADL.

One can, however, retrieve the LRM using either the *Bewley Instrumental Variable Regression* or the *Delta Method*.

A Bounds Approach to Inference using the LRM

The Bewley Method

This standard error can be estimated using the Bewley ECM. The Bewley is estimated as an instrumental variable (IV) regression.

The first (instrument) stage for an ADL(1,1;1) takes the form

$$\Delta \hat{Y}_t = \alpha_0 + \alpha_1 Y_{t-1} + \beta_0 X_1 + \beta_1 X_{t-1} + \nu_t$$

The second (endogenous) stage takes the form

$$Y_t = \phi_0 - \phi_1 \Delta \hat{Y}_t + \psi_0 X_t - \psi_1 \Delta X_t + \mu_t$$

where, $\phi_0 = \frac{\alpha_0}{1-\alpha_1}$, $\phi_1 = \frac{1}{1-\alpha_1}$, $\psi_0 = \frac{\beta_0+\beta_1}{1-\alpha_1}$, $\psi_1 = \frac{\beta_1}{1-\alpha_1}$, $\mu_t = \frac{\varepsilon_t}{1-\alpha_1}$

A Bounds Approach to Inference using the LRM

The coefficient on X_t in the second stage of the model (ψ_0) is the long run multiplier for the ADL(1,1;1).

$$\psi_0 = \frac{\beta_0 + \beta_1}{1 - \alpha_1} \qquad LRM(X_i) = \frac{\beta_0 + \beta_1}{1 - \alpha_1}$$

We can use the coefficient, standard error, and t-test from this model to test the null that the LRM(X_t) $= 0$.

Restricted versions of the Bewley ECM are used to retrieve the standard errors for the restricted models.

Type	Stage 1	Stage 2
Autoregressive Distributed Lag	$\Delta\hat{y}_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \beta_1 x_{t-1} + \nu_t$	$y_t = \phi_0 + \phi_1 \Delta\hat{y}_t + \psi_0 X_t + \psi_1 \Delta X_t + \varepsilon_t$
Partial Adjustment[a]	$\Delta\hat{y}_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_0 x_t + \nu_t$	$y_t = \phi_0 + \phi_1 \Delta\hat{y}_t + \psi_0 X_t + \varepsilon_t$
Static[b]	NA	NA
Finite Distributed Lag[c]	NA	NA
Dead Start[e]	$\Delta\hat{y}_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_1 x_{t-1} + \nu_t$	$y_t = \phi_0 + \phi_1 \Delta\hat{y}_t + \psi_1 X_{t-1} + \varepsilon_t$

A Bounds Approach to Inference using the LRM

```
> toy.adl.mod <- summary(dynlm(y.adl~ L(y.adl,1) + x + L(x,1),data=toy.data.ts))
> toy.adl.mod

Time series regression with "ts" data:
Start = 1985(2), End = 2009(12)

Call:
dynlm(formula = y.adl ~ L(y.adl, 1) + x + L(x, 1), data = toy.data.ts)

Residuals:
     Min       1Q   Median       3Q      Max
-2.50629 -0.66606 -0.04162  0.71521  2.47697

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0257453  0.0576192  -0.447    0.655
L(y.adl, 1)  0.7995721  0.0008929 895.464   <2e-16 ***
x           12.0709249  0.0552153 218.616   <2e-16 ***
L(x, 1)      4.9116005  0.0662717  74.113   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9666 on 295 degrees of freedom
Multiple R-squared:  0.9999,    Adjusted R-squared:  0.9999
F-statistic: 8.191e+05 on 3 and 295 DF,  p-value: < 2.2e-16
```

A Bounds Approach to Inference using the LRM

```
> ### ADL
> ## Calculate LRM
> b.adl <- toy.adl.mod$coefficients
> LRM.adl.X <- ((b.adl[3] + b.adl[4])/(1 - b.adl[2]))
> LRM.adl.X
[1] 85.11079
> ## SE for LRM ADL
> # Y variables for ivreg
> lagy.adl <- tslag(y.adl)
> y.diff.adl <- y.adl - tslag(y.adl)
> # Dataset for ivreg
> ivreg.data.adl <- data.frame(cbind(t,y.adl,y.diff.adl,lagy.adl,x,x.diff,lagx))
> # Specify instrument equation
> inst.adl <- cbind(1,lagy.adl,x,lagx)
> # Estimate bewley regression
> bewley.adl <- summary(ivreg(y.adl ~ y.diff.adl + x + x.diff | inst.adl, data = ivreg.data.adl))
> bewley.adl

Call:
ivreg(formula = y.adl ~ y.diff.adl + x + x.diff | inst.adl, data = ivreg.data.adl)

Residuals:
     Min       1Q   Median       3Q      Max
-16.0445  -3.7108   0.2065   3.6224  13.1254

Coefficients:
             Estimate Std. Error  t value Pr(>|t|)
(Intercept)   0.17664    0.30103    0.587    0.558
y.diff.adl   -3.97406    0.02551 -155.780   <2e-16 ***
x            85.11079    0.33548  253.700   <2e-16 ***
x.diff      -25.41784    0.30982  -82.042   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.179 on 295 degrees of freedom
Multiple R-Squared: 0.9958,     Adjusted R-squared: 0.9958
Wald test: 2.347e+04 on 3 and 295 DF,  p-value: < 2.2e-16
```

A Bounds Approach to Inference using the LRM

```
> ### PA
> ## Calculate LRM
> b.pa <- toy.pa.mod$coefficients
> LRM.pa.X <- (b.pa[3]/(1 - b.pa[2]))
> LRM.pa.X
[1] 60.24802
> ## SE for LRM PA
> # Y variables for ivreg
> lagy.pa <- tslag(y.pa)
> y.diff.pa <- y.pa - tslag(y.pa)
> # Dataset for ivreg
> ivreg.data.pa <- data.frame(cbind(y.pa,y.diff.pa,lagy.pa,x,x.diff,lagx))
> inst.pa <- cbind(1,lagy.pa,x)
> bewley.pa <- summary(ivreg(y.pa ~ y.diff.pa + x | inst.pa, data = ivreg.data.pa))
> bewley.pa

Call:
ivreg(formula = y.pa ~ y.diff.pa + x | inst.pa, data = ivreg.data.pa)

Residuals:
    Min      1Q  Median      3Q     Max 
-14.789  -3.886   0.224   3.592  12.917 

Coefficients:
            Estimate Std. Error  t value Pr(>|t|)    
(Intercept)  0.19435    0.30280    0.642    0.521    
y.diff.pa   -3.99537    0.03121 -128.035   <2e-16 ***
x           60.24802    0.32411  185.888   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.213 on 296 degrees of freedom
Multiple R-Squared: 0.9916,	Adjusted R-squared: 0.9916 
Wald test: 1.767e+04 on 2 and 296 DF,  p-value: < 2.2e-16
```

A Bounds Approach to Inference using the LRM

```
> ### Static
> ## Calculate LRM
> LRM.static <- toy.static.mod$coefficients[,1][2]
> LRM.static
       x
12.02536
> ### FDL
> ## LRM
> b.fdl <- toy.fdl.mod$coefficients
> LRM.fdl <- (b.fdl[,1][2] + b.fdl[,1][3])
> LRM.fdl
       x
17.07466
> ## SE LRM FDL (SE for Two Coefficients)
> auxreg.data.fdl <- data.frame(cbind(y.fdl,x,lagx))
> se.LRM.fdl <- sqrt((toy.fdl.mod$coefficients[,2][2])^{2} + (toy.fdl.mod$coefficients[,2][3])^{2} - 2*(vcov(lm(y.fdl~x +
lagx,data=auxreg.data.fdl))[,3][2]))
> ## F on X and Lag X against Null that X and Lag X should be omitted form the model.
> F.fdl.X <- anova(lm(y.fdl~ x + lagx,data=na.omit(auxreg.data.fdl)),lm(y.fdl~1,data=na.omit(auxreg.data.fdl)))$F[2]
> f.crit.fdl.X <- qf(.95, df1=2, df2=length(y.fdl)-length(auxreg.data.fdl$coefficients[,1]))
> F.fdl.X
[1] 80372.7
> f.crit.fdl.X
[1] 3.025847
```

A Bounds Approach to Inference using the LRM

```
> ### DS
> ## LRM DS
> b.ds <- toy.ds.mod$coefficients
> LRM.ds <- (b.ds[,1][3]/ (1- b.ds[,1][2]))
> LRM.ds
 L(x, 1)
25.09982
> ## SE LRM DS
> # Y variables for ivreg
> y.diff.ds <- y.ds - tslag(y.ds)
> lagy.ds <- tslag(y.ds)
> # Dataset for ivreg
> ivreg.data.ds <- data.frame(cbind(y.ds,y.diff.ds,lagy.ds,x,lagx,x.diff))
> inst.ds <- cbind(1,lagy.ds,lagx)
> bewley.ds <- summary(ivreg(y.ds ~ y.diff.ds + lagx | inst.ds, data = ivreg.data.ds))
> bewley.ds

Call:
ivreg(formula = y.ds ~ y.diff.ds + lagx | inst.ds, data = ivreg.data.ds)

Residuals:
     Min       1Q   Median       3Q      Max
-15.9063  -3.6711   0.1948   3.6043  13.0045

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.17689    0.29716   0.595    0.552
y.diff.ds   -3.92232    0.07215 -54.367   <2e-16 ***
lagx        25.09982    0.31769  79.008   <2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 5.116 on 296 degrees of freedom
Multiple R-Squared: 0.9537,     Adjusted R-squared: 0.9534
Wald test:  3192 on 2 and 296 DF,  p-value: < 2.2e-16
```

A Bounds Approach to Inference using the LRM

```
> LRM.out.master <- cbind(t(lrm.adl.out),t(lrm.pa.out),t(lrm.static.out),t(lrm.fdl.out),t(lrm.ds.out))
> colnames(LRM.out.master) <- c("ADL","PA","Static","FDL","DS")
> rownames(LRM.out.master) <- c("LRM","SE(LRM)","t/F")
> LRM.out.master
                ADL          PA      Static          FDL         DS
LRM      85.1107949  60.2480150  12.02535731    17.074657 25.0998184
SE(LRM)   0.3354783   0.3241089   0.05355349     0.113776  0.3176885
t/F     253.6998743 185.8881761 224.54851331 80372.702537 79.0076516
>
> xtable(LRM.out.master)
% latex table generated in R 3.3.0 by xtable 1.8-2 package
% Thu Jun 30 17:14:14 2016
\begin{table}[ht]
\centering
\begin{tabular}{rrrrrr}
  \hline
 & ADL & PA & Static & FDL & DS \\
  \hline
LRM & 85.11 & 60.25 & 12.03 & 17.07 & 25.10 \\
  SE(LRM) & 0.34 & 0.32 & 0.05 & 0.11 & 0.32 \\
  t/F & 253.70 & 185.89 & 224.55 & 80372.70 & 79.01 \\
   \hline
\end{tabular}
\end{table}
```

A Bounds Approach to Inference using the LRM

	ADL	PA	Static	FDL	DS
LRM	85.11	60.25	12.03	17.07	25.10
SE(LRM)	0.34	0.32	0.05	0.11	0.32
t/F	253.70	185.89	224.55	80372.70	79.01

A Bounds Approach to Inference using the LRM

```
import delimited "/Users/claywebb/Dropbox/KU/Courses/POLS 706 - Grad Stats/ivregadl.csv", encoding(ISO-8859-1)

**
** Bewley Specifications
**

** Note: Stata include the instrumented variable in the regression automatically
** Note: All exogenous variables have to be included in both stages

** ADL
ivreg yadl x xdiff (ydiffadl = x lagx lagyadl)

** PA
ivreg ypa x (ydiffpa = x lagyadl)

** Static
reg ystatic x
test x

** FDL
reg yfdl x lagx
test x lagx

** DS
ivreg yds lagx (ydiffpa = lagx lagyadl)
```

The Delta Method

The delta method is a general method for deriving the variances for functions of random variables with known variances.

The coefficients from the regression are, themselves, random variables. So we can use the delta method to estimate the second moment of the ratio of these variables.

For more infomration: Details on the Delta Method

We applied the Delta Method to find the standard errors in the Labour Vote Intention example from Webb, Linn, and Lebo (2020).

Interpretation

```
> ADL.t <- dynlm(mvilab~L(mvilab)+ mgvldrsat + L(mgvldrsat) + meoi
+               + L(meoi) + moppldrsat + L(moppldrsat) + trend(mvilab, scale=FALSE),data=df.ts)
> summary(ADL.t)

Time series regression with "ts" data:
Start = 1997(6), End = 2010(4)

Call:
dynlm(formula = mvilab ~ L(mvilab) + mgvldrsat + L(mgvldrsat) +
    meoi + L(meoi) + moppldrsat + L(moppldrsat) + trend(mvilab,
    scale = FALSE), data = df.ts)

Residuals:
    Min      1Q  Median      3Q     Max
-9.1003 -1.6034  0.1127  1.5176  8.0690

Coefficients:
                             Estimate Std. Error t value Pr(>|t|)
(Intercept)                  22.98564    3.48970   6.587 7.61e-10 ***
L(mvilab)                     0.39236    0.07613   5.154 8.16e-07 ***
mgvldrsat                     0.30939    0.05089   6.080 1.00e-08 ***
L(mgvldrsat)                 -0.06888    0.05850  -1.177   0.2409
meoi                         -0.00935    0.02526  -0.370   0.7119
L(meoi)                      -0.02074    0.02485  -0.835   0.4053
moppldrsat                   -0.07745    0.06720  -1.153   0.2509
L(moppldrsat)                -0.12252    0.06670  -1.837   0.0683 .
trend(mvilab, scale = FALSE) -0.01416    0.01223  -1.158   0.2487
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 2.545 on 146 degrees of freedom
Multiple R-squared:  0.9116,	Adjusted R-squared:  0.9068
F-statistic: 188.2 on 8 and 146 DF,  p-value: < 2.2e-16
```

Interpretation

```
> # Pull out LRM and use delta method to calculate standard errors ----------
>
> LRM.PM <- (ADL.t$coefficients[3]+ADL.t$coefficients[4])/(1-ADL.t$coefficients[2])
> LRM.PM
mgvldrsat
0.3958122
> deltamethod(~(x3+x4)/(x2-1),coef(ADL.t), vcov(ADL.t))
[1] 0.05152719
>
> LRM.OPP <- (ADL.t$coefficients[7]+ADL.t$coefficients[8])/(1-ADL.t$coefficients[2])
> LRM.OPP
moppldrsat
-0.3290961
> deltamethod(~(x7+x8)/(x2-1),coef(ADL.t), vcov(ADL.t))
[1] 0.06675278
>
> LRM.EO <- (ADL.t$coefficients[5]+ADL.t$coefficients[6])/(1-ADL.t$coefficients[2])
> LRM.EO
       meoi
-0.04952495
> deltamethod(~(x5+x6)/(x2-1),coef(ADL.t), vcov(ADL.t))
[1] 0.02103363
>
```

Interpretation

Table 5 ADL Model of Labour Vote Intention, May 1997–April 2010

Labour Vote Intentions	ADL	LRM x_{it}	LRM$-t$
$\alpha_1 y_{t-1}$			
Labour Vote Intention	0.392		
	(0.076)		
$\beta_0 x_t$			
Prime Minister Approval	0.309	0.396	7.68
	(0.051)	(0.052)	(Beyond)
Economic Optimism	−0.009	−0.050	−2.35
	(0.025)	(0.021)	(Between)
Leader of Opposition Approval	−0.077	−0.329	−4.93
	(0.067)	(0.067)	(Beyond)
$\beta_1 x_{t-1}$			
Prime Minister Approval	−0.069		
	(0.058)		
Economic Optimism	−0.021		
	(0.025)		
Leader of Opposition Approval	−0.123		
	(0.067)		
Trend	−0.014		
	(0.012)		
Constant	22.986		
	(3.489)		
N	155		
R^2	0.912		
Breusch-Godfrey (12 lags)	0.831		

Note: Standard errors are in parentheses. The LRM, LRM_{SE}, and t-LRM, are estimated from Equation (4), the Bewley instrumental variables regression. The t-statistics are reported as "Below" when $|t| < 1.01$, "Between" when $1.01 < |t| < 3.65$, and "Beyond" when $|t| > 3.65$.

A Bounds Approach to Inference using the LRM

The Bounds

If we have our LRM and a standard error for the LRM, we have everything we need, right? Unfortunately, no.

We have already established that the distribution for our test statistic on α_1^* can change depending on:

- The presence of a constant/drift term, c_0
- The presence of a trend, c_t
- The number of independent variables, k
- The number of observations, T
- The classification of y

All of these things can affect the distribution of t_{LRM}.

t_{LRM} also varies based on the autoregressive properties of x and y

A Bounds Approach to Inference using the LRM

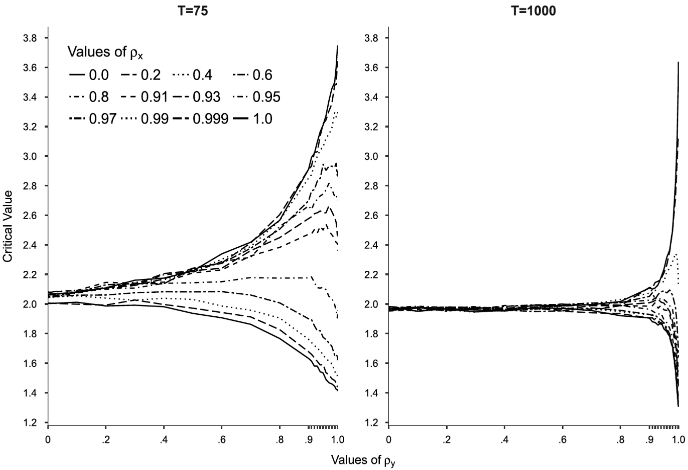


Figure 1. Simulated critical values for the LRM t-test (95th percentile). *Note:* Critical values are computed via stochastic simulations using 50,000 replications for the LRM t-statistic in the Bewley instrumental variables regression in equation (9). The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_t = \rho_x x_{t-1} + e_{xt}$ where the errors are drawn from independent standard normal distributions.

A Bounds Approach to Inference using the LRM

What do we observe?

1. The pattern is the same in both panels.
2. The critical values are standard normal for both sample sizes when y is white noise, regardless of the dynamics in x.
3. As y becomes more autoregressive, the appropriate critical values fan out based on the degree of autocorrelation in x.
4. The upper and lower limits of the critical values are the same in both panels. These are the bounds for the critical values.

Conclusion: While the critical values vary substantially within the upper and lower limits based on ρ_x and ρ_y across the two sample sizes, *the upper and lower limits are constant across sample sizes.*

We have unpredictable critical values but predictable value bounds.

A Bounds Approach to Inference using the LRM

We have observed the bounds in a single case:

$$y_t = \rho_y y_{t-1} + e_{yt} \qquad \text{and} \qquad x_t = \rho_x x_{t-1} + e_{xt}$$

We want to know how these bounds behave under other conditions:

- Different numbers of independent variables, k
- Different deterministic elements, c_0 and c_t
- Different sample sizes

If the bounds behave predictably, we can establish bounds for *specific* conditions and a *general* set of bounds.

The next slide shows the critical values for three independent variables ($k = 3$) using three independent time series.

Note: The bounds are set by the null condition. In the next slide y is not a function of x_1, x_2, or x_3.

A Bounds Approach to Inference using the LRM

Table 3. The empirical distribution of the ECM *t*-test and simulated critical values for the LRM *t*-test: identifying the bounds conditions.

	y_{t-1}		$x_{1,t-1}$		$x_{2,t-1}$		$x_{3,t-1}$	
	2.5%	**97.5%**	**2.5%**	**97.5%**	**2.5%**	**97.5%**	**2.5%**	**97.5%**
$\rho_y = 0$								
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−33.54	−29.61	−1.96	1.97	−1.96	1.97	−1.97	1.97
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−33.57	−29.65	−1.98	1.98	−1.96	1.97	−1.96	1.96
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−33.59	−29.63	−1.97	1.96	−1.97	1.97	−1.96	1.97
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−33.57	−29.63	−1.97	1.96	−1.96	1.98	−1.96	1.98
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−33.59	−29.68	−1.99	1.98	−1.96	1.97	−1.97	1.97
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−33.60	−29.67	−1.97	1.98	−1.97	1.96	−1.98	1.97
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−33.61	−29.67	−1.97	1.97	−1.97	1.98	−1.97	1.98
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−33.63	−29.69	−1.96	1.97	−1.97	1.97	−1.97	1.96
$\rho_y = 1$								
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−3.13	0.24	−1.30	1.30	−1.29	1.29	−1.30	1.29
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−3.50	0.08	−3.65	3.62	−1.37	1.37	−1.38	1.38
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−3.49	0.07	−1.38	1.37	−3.63	3.65	−1.38	1.37
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−3.49	0.07	−1.38	1.38	−1.37	1.37	−3.64	3.67
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−3.78	−0.13	−3.42	3.46	−3.44	3.42	−1.44	1.43
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−3.77	−0.13	−3.44	3.43	−1.43	1.43	−3.44	3.38
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−3.77	−0.13	−1.44	1.43	−3.41	3.43	−3.40	3.46
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−4.04	−0.33	−3.27	3.32	−3.27	3.29	−3.28	3.27

Note: Critical values are computed via stochastic simulations using 100,000 replications of $T = 1000$ for the LRM *t*-statistic in the Bewley instrumental variables regression in equation (9). A constant $\mathbf{x}_t$, $\mathbf{x}_{t-1}$, and y_{t-1} are used as instruments. The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_{i,t} = \rho_{x_i} x_{i,t-1} + e_{x_i,t}$ for $i = 1, 2, 3$, where the errors are drawn from independent standard normal distributions.

A Bounds Approach to Inference using the LRM

Table 3. The empirical distribution of the ECM t-test and simulated critical values for the LRM t-test: identifying the bounds conditions.

	y_{t-1}		$x_{1,t-1}$		$x_{2,t-1}$		$x_{3,t-1}$	
	2.5%	97.5%	2.5%	97.5%	2.5%	97.5%	2.5%	97.5%
$\rho_y = 0$								
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−33.54	−29.61	−1.96	1.97	−1.96	1.97	−1.97	1.97
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−33.57	−29.65	−1.98	1.98	−1.96	1.97	−1.96	1.96
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−33.59	−29.63	−1.97	1.96	−1.97	1.97	−1.96	1.97
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−33.57	−29.63	−1.97	1.96	−1.96	1.98	−1.96	1.98
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−33.59	−29.68	−1.99	1.98	−1.96	1.97	−1.97	1.97
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−33.60	−29.67	−1.97	1.98	−1.97	1.96	−1.98	1.97
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−33.61	−29.67	−1.97	1.97	−1.97	1.98	−1.97	1.98
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−33.63	−29.69	−1.96	1.97	−1.97	1.97	−1.97	1.96
$\rho_y = 1$								
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−3.13	0.24	−1.30	1.30	−1.29	1.29	−1.30	1.29
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 0$	−3.50	0.08	−3.65	3.62	−1.37	1.37	−1.38	1.38
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−3.49	0.07	−1.38	1.37	−3.63	3.65	−1.38	1.37
$\rho_{x1} = 0 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−3.49	0.07	−1.38	1.38	−1.37	1.37	−3.64	3.67
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 0$	−3.78	−0.13	−3.42	3.46	−3.44	3.42	−1.44	1.43
$\rho_{x1} = 1 \mid \rho_{x2} = 0 \mid \rho_{x3} = 1$	−3.77	−0.13	−3.44	3.43	−1.43	1.43	−3.44	3.38
$\rho_{x1} = 0 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−3.77	−0.13	−1.44	1.43	−3.41	3.43	−3.40	3.46
$\rho_{x1} = 1 \mid \rho_{x2} = 1 \mid \rho_{x3} = 1$	−4.04	−0.33	−3.27	3.32	−3.27	3.29	−3.28	3.27

Note: Critical values are computed via stochastic simulations using 100,000 replications of $T = 1000$ for the LRM t-statistic in the Bewley instrumental variables regression in equation (9). A constant $\mathbf{x}_t$, $\mathbf{x}_{t-1}$, and y_{t-1} are used as instruments. The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_{i,t} = \rho_{x_i} x_{i,t-1} + e_{x_i,t}$ for $i = 1, 2, 3$, where the errors are drawn from independent standard normal distributions.

A Bounds Approach to Inference using the LRM

For a given set of *deterministic components* ($c_0 = c_t = 0$), a given sample size, and a given number of independent variables ($k = 3$); the upper and lower bounds are set by the conditions where:

- **Lower Bound:** y is I(1) and all x are I(0)
  - *Intuition:* y and X are least similar.
  - Least likey that y and X will appear related.
- **Upper Bound:** y is I(1) and exactly one x is I(1)
  - *Intuition:* y and one x are very similar.
  - This increases the possibility of spurious regression.
  - The opportunity for spurious covariation among the elements of X increases as the number of I(1) k increases.
  - This covariation has the same effect as multicollinearity.

Next, we want to extend this to different numbers of independent variables (k) and different deterministic components (c_0 and c_t).

Table 5. Upper and lower bounds for the LRM t-test by k and T and deterministic components.

	$T = 75$		$T = 150$		$T = 1000$		$T = 75$		$T = 150$		$T = 1000$	
k	**LB**	**UB**	**LB**	**UB**	**LB**	**UB**	**LB**	**UB**	**LB**	**UB**	**LB**	**UB**
	Model: $c_0 \neq 0.0$ $c_1 = 0.0$ DGP: $c_0 = 0.0$ $c_1 = 0.0$						Model: $c_0 \neq 0.0$ $c_1 \neq 0.0$ DGP: $c_0 = 0.0$ $c_1 \neq 0.0$					
1	1.40	3.69	1.35	3.69	1.29	3.65	1.05	1.94	1.01	1.89	0.99	1.85
2	1.40	3.68	1.34	3.63	1.30	3.60	1.05	1.99	1.01	1.90	0.99	1.88
3	1.40	3.62	1.35	3.63	1.30	3.65	1.05	1.93	1.01	1.88	0.99	1.86
4	1.40	3.61	1.34	3.59	1.30	3.61	1.05	1.91	1.01	1.89	0.99	1.86
	Model: $c_0 \neq 0.0$ $c_1 = 0.0$ DGP: $c_0 \neq 0.0$ $c_1 = 0.0$						Model: $c_0 \neq 0.0$ $c_1 \neq 0.0$ DGP: $c_0 \neq 0.0$ $c_1 \neq 0.0$					
1	1.06	3.25	1.01	3.04	0.99	2.89	1.06	1.95	1.02	1.90	0.98	1.86
2	1.06	3.26	1.02	3.04	0.98	2.88	1.05	1.92	1.01	1.91	0.99	1.86
3	1.06	3.24	1.01	3.07	0.98	2.87	1.05	1.97	1.01	1.90	0.98	1.87
4	1.07	3.29	1.01	3.09	0.99	2.90	1.06	1.92	1.02	1.89	0.98	1.87

Note: Critical values are computed via stochastic simulations using 100,000 replications for the LRM t-statistic in the Bewley instrumental variables regression in equation (9). A constant x_t, x_{t-1}, and y_{t-1} are used as instruments. The time series y_t and x_t are generated from: $y_t = c_0 + c_1 t + \rho_y y_{t-1} + e_{yt}$ and $x_{i,t} = \rho_{x_i} x_{i,t-1} + e_{x_i,t}$ for $i = 1, 2, 3, 4$, where the errors are drawn from independent standard normal distributions. c_0 denotes the constant and c_1 the trend. The constant in the DGP (c_0) took values of 0 and 1.

A Bounds Approach to Inference using the LRM

What do we observe?

1. Within deterministic components, the upper and lower bounds are relatively stable across k and T.
2. We see variation across the lower and upper bounds based on the deterministic components.
   - The *highest upper bounds* exist for $c_0 = c_t = 0$.
   - The *lowest lower bounds* exist for $c_0 \neq$ and $c_t \neq 0$.
3. There are not major differences among the deterministics.

The results for the upper and lower bounds across conditions are, again, a result of the similarities between y and the elements of X.

These results establish conditions for a **general** set of bounds that combine the *highest upper bounds* and the *lowest lower bounds*.

A Bounds Approach to Inference using the LRM

Webb, Linn, and Lebo (2019) establish a general set of bounds.

Table 6. Bounds given uncertainty about deterministic features of the DGP.

	$T = 75$		$T = 150$		$T = 1000$	
k	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
1	1.05	3.69	1.01	3.69	0.98	3.65
2	1.05	3.68	1.01	3.63	0.98	3.60
3	1.05	3.62	1.01	3.63	0.98	3.65
4	1.05	3.61	1.01	3.59	0.98	3.61

Note: Critical values are computed via stochastic simulations using 100,000 replications for the LRM t-statistic in the Bewley instrumental variables regression in equation (9). A constant x_t, x_{t-1}, and y_{t-1} are used as instruments. The time series y_t and x_t are generated from: $y_t = \rho_y y_{t-1} + e_{yt}$ and $x_{i,t} = \rho_{x_i} x_{i,t-1} + e_{x_i,t}$ for $i = 1, 2, 3, 4$, where the errors are drawn from independent standard normal distributions.

Webb, Linn, and Lebo (2020) extend these bounds.

A Bounds Approach to Inference using the LRM

	$T=25$		$T=50$		$T=75$		$T=150$		$T=500$		$T=1000$	
k	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
$\alpha=.01$												
1	1.86	6.34	1.53	5.94	1.43	5.92	1.35	5.84	1.32	5.85	1.32	5.74
2	1.91	6.17	1.54	6.00	1.44	5.99	1.36	5.75	1.32	5.83	1.30	5.81
3	1.99	6.16	1.55	5.97	1.46	5.86	1.35	5.86	1.32	5.84	1.30	5.77
4	2.08	6.06	1.55	5.93	1.43	5.70	1.36	5.75	1.32	5.71	1.31	5.82
5	2.21	6.18	1.56	5.75	1.44	5.68	1.37	5.75	1.34	5.68	1.30	5.79
$\alpha=.05$												
1	1.25	3.85	1.09	3.68	1.04	3.68	1.02	3.65	1.00	3.66	0.99	3.62
2	1.27	3.71	1.09	3.69	1.05	3.68	1.02	3.66	0.99	3.67	0.99	3.61
3	1.30	3.63	1.10	3.56	1.07	3.62	1.01	3.62	0.99	3.62	0.99	3.61
4	1.33	3.60	1.10	3.60	1.05	3.60	1.02	3.62	0.99	3.60	0.99	3.64
5	1.40	3.55	1.11	3.53	1.07	3.56	1.02	3.56	1.00	3.60	0.99	3.62
$\alpha=.10$												
1	0.99	2.88	0.89	2.79	0.86	2.80	0.84	2.77	0.83	2.80	0.82	2.76
2	1.00	2.75	0.89	2.77	0.87	2.77	0.85	2.77	0.83	2.76	0.83	2.76
3	1.01	2.71	0.90	2.69	0.87	2.73	0.84	2.75	0.83	2.77	0.83	2.74
4	1.04	2.67	0.90	2.71	0.87	2.71	0.84	2.74	0.83	2.75	0.83	2.75
5	1.08	2.57	0.90	2.64	0.88	2.67	0.84	2.73	0.84	2.73	0.83	2.76

A Bounds Approach to Inference using the LRM

Applying the Bounds

The bounds are easy to use.

Step 1: Specify a dynamic model.

Step 2: Calculate the LRM and the SE(LRM).

Step 3: Calculate $t_{LRM} = \frac{LRM}{SE(LRM)}$.

Step 4: Compare t_{LRM} to the bounds.

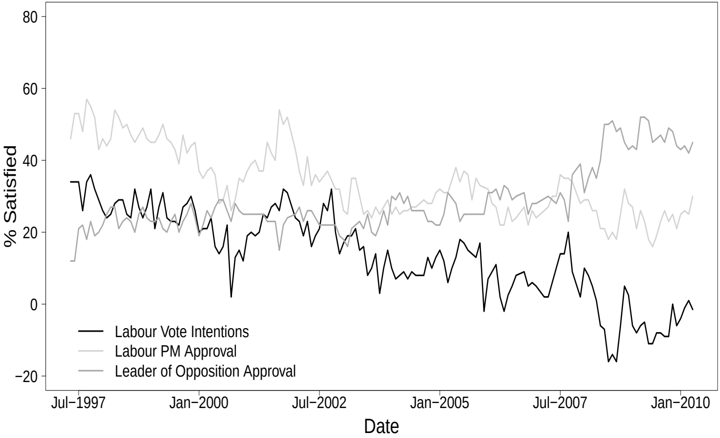
Interpretation

- **Below:** Fail to reject the null.
- **Beyond:** Reject the null.
- **Between:** Uncertain

The area of indeterminacy may seem unsatisfying, but this uncertainty was always present. Now the uncertainty is explicit.

A Bounds Approach to Inference using the LRM

Figure: Vote Intentions for Labour, Prime Ministerial Approval, and Leader of Opposition Approval, May 1997-April 2010



A Bounds Approach to Inference using the LRM

Table 5 ADL Model of Labour Vote Intention, May 1997–April 2010

Labour Vote Intentions	ADL	LRM x_{it}	LRM−t
$\alpha_1 y_{t-1}$			
Labour Vote Intention	0.392		
	(0.076)		
$\beta_0 x_t$			
Prime Minister Approval	0.309	0.396	7.68
	(0.051)	(0.052)	(Beyond)
Economic Optimism	−0.009	−0.050	−2.35
	(0.025)	(0.021)	(Between)
Leader of Opposition Approval	−0.077	−0.329	−4.93
	(0.067)	(0.067)	(Beyond)
$\beta_1 x_{t-1}$			
Prime Minister Approval	−0.069		
	(0.058)		
Economic Optimism	−0.021		
	(0.025)		
Leader of Opposition Approval	−0.123		
	(0.067)		
Trend	−0.014		
	(0.012)		
Constant	22.986		
	(3.489)		
N	155		
R^2	0.912		
Breusch-Godfrey (12 lags)	0.831		

Note: Standard errors are in parentheses. The LRM, LRM_{SE}, and t-LRM, are estimated from Equation (4), the Bewley instrumental variables regression. The t-statistics are reported as "Below" when $|t| < 1.01$, "Between" when $1.01 < |t| < 3.65$, and "Beyond" when $|t| > 3.65$.

Example - Wolak and Peterson (2020)

AJPS AMERICAN JOURNAL of POLITICAL SCIENCE

The Dynamic American Dream

Jennifer Wolak University of Colorado
David A. M. Peterson Iowa State University

Abstract: *The American Dream is central to the national ethos, reflecting people's optimism that all who are willing to work hard can achieve a better life than their parents. Separate from the support for the idea of the American Dream itself is whether the public believes it is attainable. We consider the origins and dynamics of the public's belief in the achievability of the American Dream. Is the American Dream a symbolic vision, rooted in political socialization rather than contemporary politics? Or does optimism about the American Dream follow from the viability of the dream, rising with economic prosperity and falling with declining opportunity? We develop a new macrolevel measure of belief in the American Dream from 1973 to 2018. We show that it moves over time, responsive to changes in social mobility, income inequality, and economic perceptions. As inequality increases, belief in the attainability of the American Dream declines.*

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Example - Wolak and Peterson (2020)

FIGURE 1 The Dynamics of Public Belief in the American Dream

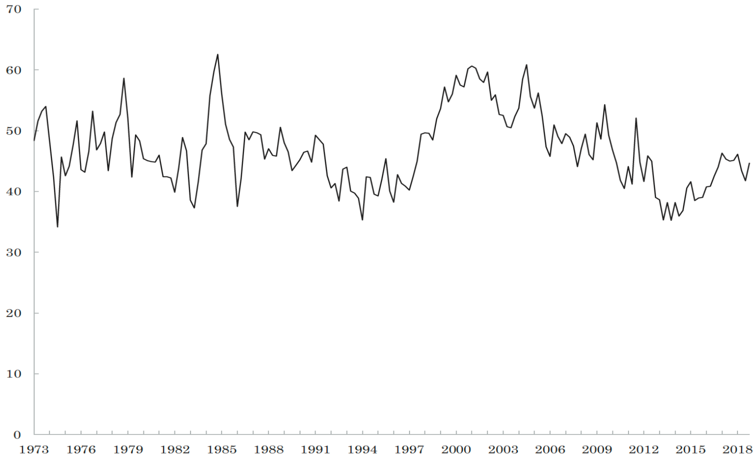


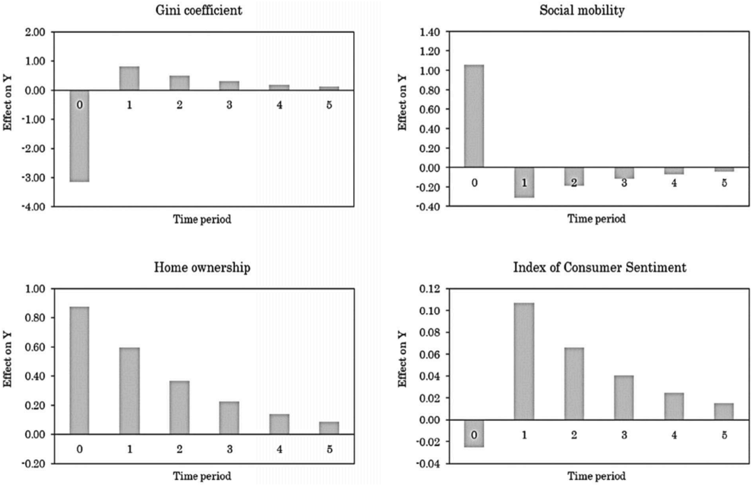
Table 1 Explaining Belief in the American Dream

	Δ Belief in the American Dream	
	Model 1	Model 2
Belief in the American Dream $_{t-1}$	−0.386*	−0.388*
	(0.057)	(0.058)
Δ Gini coefficient	−315.981	—
	(251.617)	
Gini coefficient $_{t-1}$	−41.727*	—
	(11.932)	
Δ Social mobility	—	105.564
		(61.247)
Social mobility $_{t-1}$	—	10.237*
		(3.019)
Δ Homeownership	0.877	0.827
	(0.865)	(0.916)
Homeownership $_{t-1}$	0.932*	0.788*
	(0.230)	(0.219)
Δ Policy mood	−0.146	−0.116
	(0.143)	(0.152)
Policy mood $_{t-1}$	0.092	0.121
	(0.069)	(0.071)
Δ Index of consumer sentiment	−0.025	−0.032
	(0.050)	(0.051)
Index of consumer sentiment $_{t-1}$	0.097*	0.088*
	(0.024)	(0.025)
Midterm election	1.203	1.204
	(1.043)	(1.066)
Presidential campaign	0.535*	0.382
	(0.221)	(0.221)
Constant	−38.657*	−54.838*
	(12.240)	(15.443)
Long run multipliers		
LRM, Gini coefficient	−108.152†	—
standard error	(26.957)	
t-value	−4.012	
LRM, Social mobility	—	26.358†
standard error		(6.923)
t-value		3.807
LRM, Home ownership	2.416†	2.030†
standard error	(0.456)	(0.457)
t-value	5.304	4.440
LRM, Policy mood	0.239	0.311
standard error	(0.175)	(0.179)
t-value	1.368	1.737
LRM, Index of consumer sentiment	0.253†	0.226†
standard error	(0.058)	(0.060)
t-value	4.372	3.764
R^2	0.266	0.270
N	175	167
Box-Ljung Q Test	44.722	37.505
p-value	0.280	0.583

Standard errors in parentheses.
*$p < 0.05$. The p-values for the coefficients are based on their t-statistics.
†indicates significant LRMs, where the t-statistic exceeds the absolute value of 3.560.

Example - Wolak and Peterson (2020)

FIGURE 2 Estimated Lag Distributions for Belief in the American Dream



A Bounds Approach to Inference using the LRM

There are costs and benefits to this approach.

The costs to the bounds approach:

- The area of indeterminacy
- Uncertainty about the type of equilibrium

The benefits to the bounds approach:

- Classification uncertainty is reflected in the hypothesis test.
- Our inferences do not rely on unit root tests.

Pre-testing is still an important part of time series.

If we can classify our series, we can use the information to interpret the the relationships in the data.

But ...

A Bounds Approach to Inference using the LRM

A Bounds Approach to Inference using the LRM

For more information:

Webb, Clayton, Suzanna Linn, and Matt Lebo. 2019. "A Bound Approach to Inference Using the Long Run Multiplier." *Political Analysis* 27(3): 281-301.

Webb, Clayton, Suzanna Linn, and Matt Lebo. 2020. "Beyond the Unit Root Question: Uncertainty and Inference." *American Journal of Political Science* 64(2): 275-292.

Replication materials for both papers can be found at `https://www.claytonmwebb.com`